

Deformed gas of p, q -bosons: virial expansion and virial coefficients

A.M. Gavrilik¹, A.P. Rebesh

Bogolyubov Institute for Theoretical Physics, Kiev, Ukraine

Abstract

In the study of many-particle systems both the interaction of particles can be essential and such feature as their internal (composite) structure. To describe these aspects, the theory of deformed oscillators is very efficient. Viewing the particles as p, q -deformed bosons, in the corresponding p, q -Bose gas model we obtain in explicit form virial expansion along with the 2nd to 5th virial coefficients. The obtained virial coefficients depend on the deformation parameters p, q in the form symmetric under $p \leftrightarrow q$, and at $p \rightarrow 1, q \rightarrow 1$ turn into those known for usual bosons. Besides real parameters, we analyze the case of complex mutually conjugate p and q and find interesting implications. Also, the critical temperature is derived (for the p, q -Bose gas) and compared with the T_c of standard case of bosons condensation. Similar results are presented for the deformed Bose gas model of the Tamm-Dancoff type.

Keywords: deformed Bose gas model, virial expansion, virial coefficients, critical temperature

1 Introduction

Diverse models of deformed Bose gas, elaborated on the base of a set of some deformed oscillators or deformed bosons, appeared in the literature from early nineties, see e.g. Refs. [1]-[8]. Some analysis of a number of works on this topic has been done in Ref. [9]. Since then, the statistical mechanics of a gas of q - or p, q -bosons was extensively explored (and further extended) as witnessed by [10]-[19] and many others.

With the aim of treating possible intermediate statistics behavior of physical system, one can use the method based on a deformation of quantum algebra of creation, annihilation and number operators. Deformed oscillator algebra is a generalization of the harmonic oscillator or Heisenberg algebra. Besides intermediate statistics, there exist other convincing reasons of why one should use some model of deformed oscillator instead of the usual quantum harmonic oscillator. The models of deformed oscillators are efficient in description of interacting particles system, or if the finite proper volume of particles, or their composite nature (internal structure) are to be taken into account. The theory of q -oscillators is related to the theory of quantum groups as originally shown in Ref. [20] and Ref. [21]. As a direct generalization of q -oscillators, in Ref. [22] and Ref. [23] the two-parameter family of p, q -oscillators has been introduced.

The two-parameter p, q -deformed analog of Bose gas model can be developed, see [12], [13], using the set of p, q -deformed oscillators. Some further aspects of that model were studied in the works [14], [15]. In our present work we explore the thermodynamics of (some version of) p, q -Bose gas model. In order to obtain certain thermodynamical quantities of the deformed Bose gas we use the two-parametric generalization of q -calculus realized by replacing the ordinary derivative with the p, q -analog of Jackson derivative (the latter is known as q -deformed analog of usual derivative).

¹omgavr@bitp.kiev.ua

For the p, q -deformed Bose gas we investigate two regimes: high temperatures and small densities ($\lambda^2/v \ll 1$), or low temperatures and large densities ($\lambda^2/v \gg 1$). In the first case we obtain explicitly the virial expansion of the equation of state and find new virial coefficients in addition to few known ones (mainly in more special cases of one-parameter deformed Bose gas models [7], [18]). The virial coefficients reflect effective inter-particle interactions and in the considered case depend explicitly on the deformation parameters p and q . From our formulas, the ordinary boson gas results can be recovered in the corresponding limit $p \rightarrow 1, q \rightarrow 1$. On the other hand, setting p and q as complex conjugates: $p = \bar{q} = r e^{i\theta}$, we gain yet another presentation of the virial coefficients. In that case we examine the behavior of the difference between deformed coefficients and their ordinary non-deformed prototypes (corresponding to *non-deformed* Bose gas) as function of the new parameter θ , at fixed values of the parameter r (the modulus). The situation for which Bose-Einstein condensation does occur in the considered two-parameter generalized boson gas is studied when $\lambda^2/v \gg 1$, and in this case the critical temperature $T_c^{(p,q)}$ of the condensation of deformed bosons is obtained. Comparing this with the T_c of usual Bose gas we deduce that the critical temperature of the p, q -Bose gas is larger than the critical temperature of usual bosons, for the considered range of parameters. The dependence of the ratio $T_c^{(p,q)}/T_c$ on the deformation parameters p, q is explicitly studied, visualized by the corresponding figures and the distinction from the results in Ref. [15] indicated. We also consider similar results for the q -deformed Bose gas model of Tamm-Dancoff (TD) type, which constitutes a distinguished special case.

2 Deformed oscillator models

In the theory of deformed oscillators it is convenient to use the concept [11] of deformation structure function $\varphi(N)$ which determines the particular deformed oscillator model described by its respective oscillator algebra. With $\varphi(N) \equiv a^\dagger a$, the generating elements a, a^\dagger, N of the algebra obey the relations

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad (1)$$

$$aa^\dagger - a^\dagger a = \varphi(N+1) - \varphi(N), \quad aa^\dagger = \varphi(N+1). \quad (2)$$

In deformed Fock space the ground state vector obeys usual relations

$$a|0\rangle = 0, \quad N|0\rangle = 0, \quad \langle 0|0\rangle = 1, \quad (3)$$

and for the n -particle excited state in this $\varphi(n)$ -extended Fock space we have

$$N|n\rangle = n|n\rangle, \quad \varphi(N)|n\rangle = \varphi(n)|n\rangle, \quad (4)$$

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{\varphi(n)!}}|0\rangle, \quad \varphi(n)! = \varphi(n)\varphi(n-1)\dots\varphi(1), \quad \varphi(0)! = 1. \quad (5)$$

Operators a^\dagger, a act on the state $|n\rangle$ according to

$$a^\dagger|n\rangle = \sqrt{\varphi(n+1)}|n+1\rangle, \quad a|n\rangle = \sqrt{\varphi(n)}|n-1\rangle. \quad (6)$$

In this paper we deal with the system of so-called p, q -bosons (p, q -deformed oscillators) for which the deformation structure function $\varphi(n)$ in (2)-(6) is

$$\varphi_{p,q}(n) = \frac{p^n - q^n}{p - q} \equiv [n]_{p,q}, \quad (7)$$

with $[n]_{p,q}$ being the p, q -number corresponding to a number n . Let us note that from the two-parameter family one infers a plenty of different one-parameter (q -deformed) models by imposing different relations $p = f(q)$, see [26] for more details.

3 Thermodynamics of the gas of deformed bosons

In this Section we deal with a particular variant of the p, q -deformed Bose gas model. In the grand canonical ensemble, the Hamiltonian of the system is taken as

$$H = \sum_i (\varepsilon_i - \mu) N_i \quad (8)$$

where ε_i is the (kinetic) energy of particle in the state labeled by i , N_i is the boson number operator relative to ε_i and μ is the chemical potential. One-particle non-relativistic energies are given as $\varepsilon_i = p_i^2/2m$ where $p \equiv |\vec{p}|$. Thermal averages can be calculated using the usual formulas of quantum statistical mechanics. Therefore, the thermal average of an operator A is given by the standard formula

$$\langle \hat{A} \rangle = \text{Tr}(\rho \hat{A}), \quad \rho = \frac{e^{-\beta \hat{H}}}{Z} \quad (9)$$

where ρ is the equilibrium statistical operator, $Z = Z(z)$ is the partition function,

$$Z = \text{Tr}(e^{-\beta \hat{H}}). \quad (10)$$

Besides, $\beta = 1/kT$, T is the temperature; the Boltzmann constant k is put $k = 1$.

To develop thermodynamics of the gas of deformed bosons we start with the logarithm of grand partition function

$$\ln Z = - \sum_i \ln(1 - z e^{-\beta \varepsilon_i}) \quad (11)$$

where $z = e^{\beta \mu}$ is the fugacity. For obtaining thermodynamic functions of q -deformed Bose gas the usual derivative is replaced by the Jackson or q -derivative [24]

$$\frac{d^{(q)}}{dx} f(x) = \frac{f(qx) - f(x)}{qx - x} \equiv \mathcal{D}_x^{(q)} f(x). \quad (12)$$

Note that in the limit $q \rightarrow 1$ the usual derivative is recovered from the q -derivative.

For the algebra given by (1), (2) and $\varphi(n)$ from (7) we use the two-parameter p, q -generalization of Jackson derivative² being a direct p, q -extension of (12), i.e.

$$\mathcal{D}_x^{(p,q)} f(x) = \frac{f(px) - f(qx)}{px - qx}, \quad \mathcal{D}_x^{(p,q)} x^n = \frac{p^n x^n - q^n x^n}{px - qx} = [n]_{p,q} x^{n-1}. \quad (13)$$

²Let us note that in Refs. [14], [15] yet another two-parameter analog $\tilde{D}_x^{(p,q)}$ of the Jackson derivative is exploited which is related with $D_x^{(p,q)}$ in (13) as follows: $\tilde{D}_x^{(p,q)} = \frac{p-q}{\ln p - \ln q} D_x^{(p,q)}$.

When $p = 1$, the latter p, q -extension reduces to the Jackson derivative (12).

Then, the total number of particles in the p, q -deformed theory is determined as

$$N = z \mathcal{D}_z^{(p,q)} \ln Z = \frac{1}{p-q} \left(\sum_i \ln(1 - qze^{-\beta\varepsilon_i}) - \sum_i \ln(1 - pze^{-\beta\varepsilon_i}) \right). \quad (14)$$

For large volume $V \rightarrow \infty$ and large number of particles N , we replace summation over level i by integration over the 3-momentum k . Note that the two sums in (14) diverge at $z = 1$ and $p \rightarrow 1, q \rightarrow 1$ when $\varepsilon_i = 0$. That is, these $\varepsilon_i = 0$ terms can be as large as all the rest of terms in the sums. For that reason the $\varepsilon_i = 0$ term is isolated, while the remaining two sums are replaced by integrals:

$$\sum_i \rightarrow \frac{V}{(2\pi\hbar)^3} \int d^3k. \quad (15)$$

Going over in \vec{k} -space to spherical coordinates $\varphi, \theta, k = (\vec{k} \cdot \vec{k})^{1/2}$, the integration is performed with the result given through polylogarithms. Since the latter can be presented in the form of series, we derive the result

$$N = \frac{V}{\lambda^3} \left[\frac{1}{p-q} \left(\sum_{r=1}^{\infty} \frac{(zp)^r}{r^{5/2}} - \sum_{r=1}^{\infty} \frac{(zq)^r}{r^{5/2}} \right) \right] + \frac{1}{p-q} \ln \left(\frac{1-qz}{1-pz} \right) = \frac{V}{\lambda^3} \sum_{r=1}^{\infty} \frac{[r]_{p,q}}{r^{5/2}} z^r + n_0 \quad (16)$$

where $n_0 = \frac{1}{p-q} \ln \left(\frac{1-qz}{1-pz} \right)$ and $\lambda = \sqrt{\frac{2\pi\hbar^2}{mT}}$ is the thermal wavelength. Now rewrite Eq. (16) as

$$\frac{1}{v} = \frac{1}{\lambda^3} \tilde{g}_{3/2}(z; p, q) + \frac{n_0}{V} \quad (17)$$

where $v = V/N$, and the general formula for the (p, q) -extended function \tilde{g}_n reads:

$$\tilde{g}_n(z; p, q) = \frac{1}{p-q} \left(\sum_{r=1}^{\infty} \frac{(zp)^r}{r^{n+1}} - \sum_{r=1}^{\infty} \frac{(zq)^r}{r^{n+1}} \right) = \sum_{r=1}^{\infty} \frac{[r]_{p,q} z^r}{r^{n+1}}. \quad (18)$$

This function is nothing but p, q -deformed generalization of the $g_n(z)$ function known say from Ref. [25]:

$$g_n(z) \equiv \sum_{l=1}^{\infty} \frac{z^l}{l^n}. \quad (19)$$

The function (18) is real for real p and q . We assume $0 < p \leq 1, 0 < q \leq 1$. In what follows we will use $\tilde{g}_{3/2}$ and $\tilde{g}_{5/2}$. Namely, we will exploit (for small z) their power series expansions, say up to z^5 order.

Then the thermodynamic relation $PV/T = \ln Z$ can be rewritten as

$$\frac{P}{T} = \frac{1}{\lambda^3} \tilde{g}_{5/2}(z; p, q) + \frac{1}{V} \ln(1-z). \quad (20)$$

3.1 Virial expansion of the equation of state

In the case of high temperatures and small densities ($\lambda^3/v \ll 1$), the average distance $v^{1/3}$ between particles is much larger than the thermal wavelength λ . In this case the quantum effects, the n_0 , and the second term in (20) can be viewed as negligibly small. Then, if the temperature of the gas of p, q -deformed bosons is high, so that $T > T_c^{(p,q)}$ (see Sect. 4 below for $T_c^{(p,q)}$), from (17) and (20) we find

$$\frac{1}{v} = \frac{1}{\lambda^3} \tilde{g}_{3/2}(z; p, q), \quad \frac{Pv}{T} = \frac{v}{\lambda^3} \tilde{g}_{5/2}(z; p, q). \quad (21)$$

From the first equation in (21) with account of (18) we obtain

$$\frac{\lambda^3}{v} = z + \frac{[2]_{p,q}}{2^{5/2}} z^2 + \frac{[3]_{p,q}}{3^{5/2}} z^3 + \frac{[4]_{p,q}}{4^{5/2}} z^4 + \frac{[5]_{p,q}}{5^{5/2}} z^5 + \dots \quad (22)$$

Inverting the latter equation yields

$$\begin{aligned} z = \frac{\lambda^3}{v} - \frac{[2]_{p,q}}{2^{5/2}} \left(\frac{\lambda^3}{v} \right)^2 + \left(\frac{[2]_{p,q}^2}{2^4} - \frac{[3]_{p,q}}{3^{5/2}} \right) \left(\frac{\lambda^3}{v} \right)^3 - \left(\frac{5[2]_{p,q}^3}{2^{15/2}} - \frac{5[2]_{p,q}[3]_{p,q}}{2^{5/2}3^{5/2}} + \frac{[4]_{p,q}}{2^5} \right) \left(\frac{\lambda^3}{v} \right)^4 + \\ + \left(\frac{7[2]_{p,q}^4}{2^9} + \frac{7[2]_{p,q}^2[3]_{p,q}}{2^5 3^{3/2}} - \frac{[3]_{p,q}^2}{3^4} + \frac{3[2]_{p,q}[4]_{p,q}}{2^{13/2}} + \frac{[5]_{p,q}}{5^{3/2}} \right) \left(\frac{\lambda^3}{v} \right)^5. \end{aligned} \quad (23)$$

From the second equation in (21), with $\tilde{g}_{5/2}(z; p, q)$ from (18) taken in the form expanded in z , by the use of (23) we derive for the equation of state of the deformed p, q -Bose gas the desired virial expansion

$$\frac{Pv}{T} = 1 + A \left(\frac{\lambda^3}{v} \right) + B \left(\frac{\lambda^3}{v} \right)^2 + C \left(\frac{\lambda^3}{v} \right)^3 + D \left(\frac{\lambda^3}{v} \right)^4 + \dots \quad (24)$$

where the virial coefficients from 2nd to 5th read:

$$\begin{aligned} A = -\frac{[2]_{p,q}}{2^{7/2}}, \quad B = \frac{[2]_{p,q}^2}{2^5} - \frac{2[3]_{p,q}}{3^{7/2}}, \quad C = \frac{[2]_{p,q}[3]_{p,q}}{2^{5/2}3^{3/2}} - \frac{3[4]_{p,q}}{2^7} - \frac{5[2]_{p,q}^3}{2^{17/2}}, \\ D = \frac{7[2]_{p,q}^4}{2^{10}} - \frac{[2]_{p,q}^2[3]_{p,q}}{2^4 3^{5/2}} + \frac{[2]_{p,q}[4]_{p,q}}{2^{11/2}} + \frac{2[3]_{p,q}^2}{3^5} - \frac{4[5]_{p,q}}{5^{7/2}}. \end{aligned}$$

Note, in the just obtained expressions for virial coefficients, p and q appear only through (p, q) -numbers $[2]_{p,q}$, $[3]_{p,q}$, etc. Due to that, the exchange $p \leftrightarrow q$ symmetry remains intact. Let us note that the above expressions for A , B , C , D yield the results for the particular cases of q -bosons. Namely, at $p = 1$ these reduce to the virial coefficients of the AC type q -Bose gas, and putting $p = q^{-1}$ yields the BM type q -Bose gas virial expansions and virial coefficients. Another distinguished TD type case is considered separately in Sec. 5 below. Also, it should be noted that by putting $p = f(q)$ where $f(q)$ is some fixed function, as considered in Ref. [26], we readily obtain the relevant results (virial coefficients) for the corresponding, to this choice of $f(q)$, version of q -deformed Bose gas model.

Remark that the one-parameter (q -deformed) version of the above A , B , C was given in [18], [7]. The p, q -deformed 3-rd, 4-th and 5-th virial coefficients B , C and D are new. Note also that our $A = A(p, q)$ differs from the respective coefficient in [14]. In the limiting no-deformation case $p = q = 1$ from the deformed virial coefficients A , B , C , D one recovers virial coefficients A_0 , B_0 , C_0 , D_0 of the usual Bose gas:

$$A \xrightarrow{p=q=1} A_0 = -\frac{1}{2^{5/2}}, \quad C \xrightarrow{p=q=1} C_0 = \frac{1}{2^{3/2}3^{1/2}} - \frac{3}{2^5} - \frac{5}{2^{11/2}},$$

$$B \xrightarrow{p=q=1} B_0 = \frac{1}{8} - \frac{2}{3^{5/2}}, \quad D \xrightarrow{p=q=1} D_0 = \frac{7}{2^6} - \frac{1}{2^2 \cdot 3^{3/2}} + \frac{1}{2^{5/2}} + \frac{2}{3^3} - \frac{4}{5^{5/2}}.$$

It is seen that A_0 , B_0 and C_0 coincide with the well-known virial coefficients of Bose gas given in textbooks [27], [28], as it should.

3.2 Complex parameters of deformation and virial coefficients

Now take the deformation parameters p, q as complex, mutually conjugate ones:

$$p = re^{i\theta}, \quad q = re^{-i\theta}. \quad (25)$$

Then, instead of p and q we have the parameters r and θ in terms of which the coefficients of virial expansion (24) take the form:

$$\begin{aligned} \tilde{A} &= -\frac{r \cos \theta}{2^{5/2}}, & \tilde{B} &= \frac{r^2 \cos^2 \theta}{2^3} - \frac{2r^2(2\cos 2\theta + 1)}{3^{7/2}}, \\ \tilde{C} &= \frac{r^3 \cos \theta (2\cos 2\theta + 1)}{2^{3/2} 3^{3/2}} - \frac{3r^3(\cos 3\theta + \cos \theta)}{2^6} - \frac{5r^3 \cos^3 \theta}{2^{11/2}}, \\ \tilde{D} &= \frac{7r^4 \cos^4 \theta}{2^6} - \frac{r^4 \cos^2 \theta (2\cos 2\theta + 1)}{2^2 \cdot 3^{5/2}} + \frac{r^4 \cos \theta (\cos 3\theta + \cos \theta)}{2^{7/2}} - \\ &\quad - \frac{2r^4(2\cos 2\theta + 1)^2}{3^5} - \frac{4r^4(2\cos 4\theta + 2\cos 2\theta + 1)}{5^{7/2}}. \end{aligned}$$

Note, at $r = 1$ these formulas again reduce to respective results for the q -Bose gas of BM type q -bosons, with the phase like deformation parameter $q = e^{i\theta}$.

Physical meaning of the complex deformation parameters (25) can be commented as follows. If one deals with the usual non-ideal Bose gas, the virial coefficients contain terms responsible for the (two-particle, three-particle etc.) effective interactions. When dealing along the same lines with the gas of deformed bosons (deformed Bose gas), we gain that the substructure of particles or additional inter-particle interaction is effectively taken into account. Then the first *non-trivial* virial coefficient A or \tilde{A} reflects modified two-particle interaction, the second one B or \tilde{B} involves modified three-particle interaction and so on. With two parameters r and θ at hands we might hope it is possible to diminish to zero two chosen types of interaction, say, the two-particle interaction together with the three-particle one (this would happen if $\tilde{A} = 0$ and $\tilde{B} = 0$ simultaneously). However, the parameter r (the modulus) turns out to be of no help for that aim. So, using the remaining parameter θ we can find its value(s) for which only one of the two: \tilde{A} or \tilde{B} can be made zero. If that happens say for \tilde{A} , we may conclude: at this value of θ we encounter mutual compensation of the two-particle interaction present in the usual non-ideal Bose gas against the additional contribution due to deformation, i.e. due to the physical reason (finite proper volume or substructure of particles) effectively taken into account by the deformation.

On the other hand, we can find the measure of deviation of "deformed coefficients" from the known virial coefficients [27], [28] of the standard Bose gas:

$$\alpha = \tilde{A} - A_0, \quad \beta = \tilde{B} - B_0, \quad \gamma = \tilde{C} - C_0, \quad \delta = \tilde{D} - D_0. \quad (26)$$

In Fig. 1 we plot the dependence of $\alpha, \beta, \gamma, \delta$ defined in (26) on the parameter θ at some fixed values of the second parameter r .

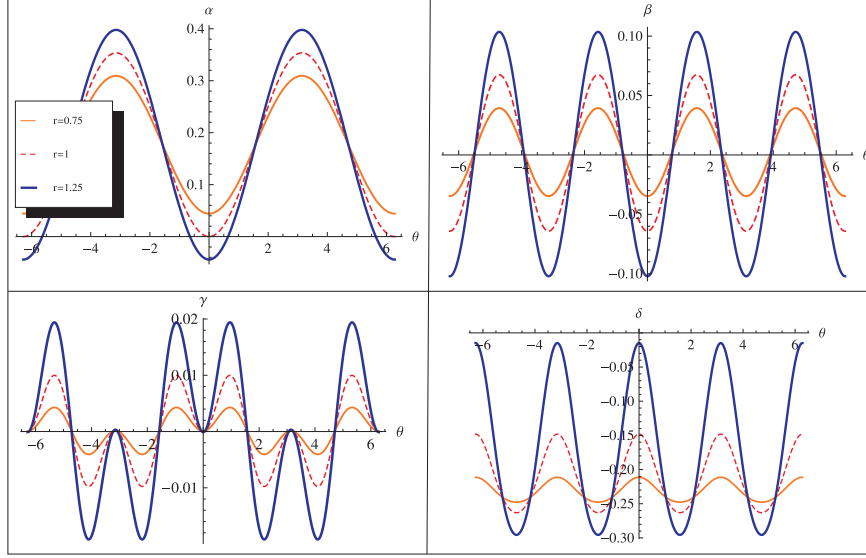


Figure 1: Dependence of α , β , γ , δ on the parameter θ at $r = 0.75, 1, 1.25$.

As seen, for different values of θ the differences α , β and γ can be positive, negative or zero, whereas the difference δ is always negative (i.e., the net contribution to the effective five-particle interaction due to deformation is smaller than the interaction in the non-deformed Bose gas).

In Fig. 2 we plot the dependence of α , β , γ , δ on the parameter θ at fixed value $r = 1$ (that implies the deformed model of BM type q -bosons).

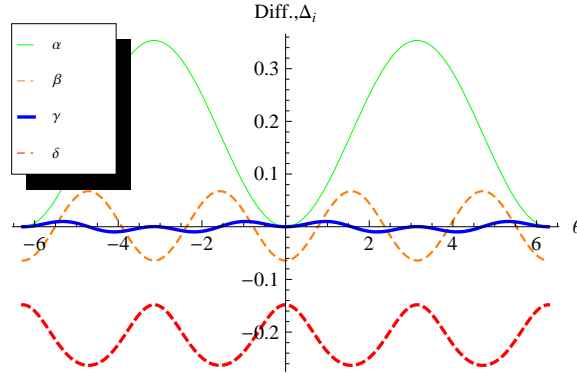


Figure 2: Dependence of the differences $\Delta_i = (\alpha, \beta, \gamma, \delta)$ from Eq. (26) on the parameter θ at $r = 1$.

Remark. From Figs 1, 2 we observe remarkable thing: for all $\theta \neq 0$, we have $\tilde{A} > A_0$, but $\tilde{D} < D_0$. That is the effect of deformation significantly increases attractive two-particle interaction for most of values of parameter θ (at $r \leq 1$ and $\theta \neq 0$). This is in some accord with remark in [10] for the case of quon-based deformation of Bose gas. In the domain of five-particle interaction the picture is opposite - the deformation modifies the interaction towards weakening. Although this regime (of high temperatures) is opposite to that one where the critical T_c is actual (see next Section), we nevertheless note that the deformation better promotes Bose condensation that results in increasing T_c (due to the increase of attractive two-particle interaction). This fact may be of potential applied value.

4 Bose condensation in p, q -Bose gas and critical temperatures

Here we study the opposite situation (to the Subsection 3.1), namely, the case of low temperatures and large densities ($\lambda^3/v \gg 1$). Eq. (21) gives the equation of state for p, q -deformed Bose gas consisting of N non-relativistic particles of mass m confined in the volume V . To explore the equation of state in detail we have to find the fugacity z as a function of temperature, and the specific volume should be found by solving the equation (17) that involves besides $n_0 = \frac{1}{p-q} \ln\left(\frac{1-qz}{1-pz}\right)$, also $v = V/N$ and the thermal wavelength $\lambda = \sqrt{2\pi\hbar^2/mT}$. Rewrite Eq. (17) as

$$\lambda^3 \frac{n_0}{V} = \frac{\lambda^3}{v} - \tilde{g}_{3/2}(z; p, q). \quad (27)$$

We see that the value n_0/V is positive if the temperature and specific volume obey the following inequality:

$$\frac{\lambda^3}{v} > \tilde{g}_{3/2}(1; p, q). \quad (28)$$

In other words, the large though finite number of particles occupies the lowest energy level with $\varepsilon_i = 0$ (ground state). That is, the phenomenon of Bose condensation takes place. For given specific volume v , the critical temperature can be found from the equality

$$\lambda_c^3 = v \tilde{g}_{3/2}(1; p, q). \quad (29)$$

Then, the critical temperature $T_c^{(p,q)}$ of the p, q -deformed Bose gas results as

$$T_c^{(p,q)} = \frac{2\pi\hbar^2/m}{[v \tilde{g}_{3/2}(1; p, q)]^{2/3}}. \quad (30)$$

Moreover, we obtain the relation between the critical temperature of the considered p, q -deformed Bose gas and the T_c of usual gas of bosons in the form of the ratio:

$$\frac{T_c^{(p,q)}}{T_c} = \left(\frac{2.61}{\tilde{g}_{3/2}(1; p, q)} \right)^{2/3}. \quad (31)$$

Note that in the no-deformation limit $p \rightarrow 1, q \rightarrow 1$, the function $\tilde{g}_{3/2}(1; p, q)$ in (30) goes over into $g_{3/2}(1) = \zeta(\frac{3}{2}) \cong 2.61$, as seen from (19) at $z = 1$. Due to this, the critical temperature $T_c^{(p,q)}$ of deformed Bose gas reduces to the critical temperature T_c of usual, non-deformed Bose gas and thus the ratio is $(T_c^{(p,q)}/T_c)|_{p=q=1} = 1$.

In Fig. 3 (*left*) we give the plot of (31) as a function of the deformation parameters p, q such that $p \leq 1, q \leq 1$. This (convex upwards) behavior with respect to p and q differs from that in [15]. Besides, this ratio increases with increasing (in the both parameters p, q) extent of deformation measured by $1 - p$ and $1 - q$. In analogy with the above case of high temperature and low density, where we considered both real p, q and the variant with complex deformation parameters $p = \bar{q} = re^{i\theta}$, here in Fig. 3 (*right*) we also present the picture for $(T_c^{(r,\theta)}/T_c)$ depending on complex p, q through the modulus r and the phase θ .

As seen, the critical temperature $T_c^{(p,q)}$ for the p, q -deformed Bose gas is larger than the critical temperature T_c for the non-deformed boson gas, at least in the chosen region $p \leq 1, q \leq 1$ of the deformation parameters (see also Remark ending Sect. 3). This fact may play an important role in future analysis involving real gases. Even more interesting is the picture for the ratio in Eq. (31) (at $p = \bar{q} = re^{i\theta}$ versus the parameters r and θ).

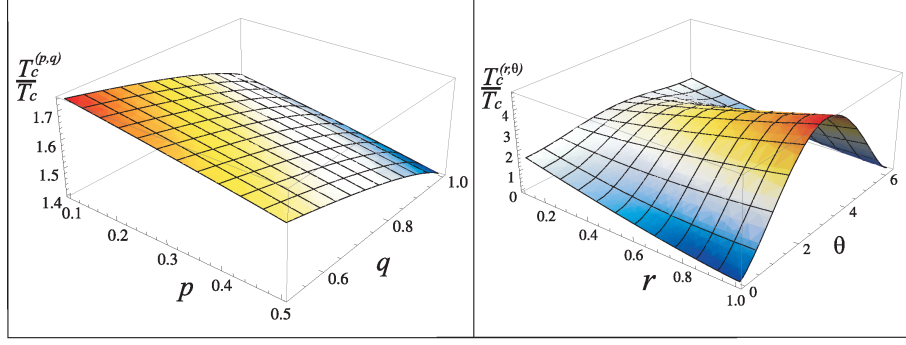


Figure 3: *Left:* The ratio $T_c^{(p,q)}/T_c$ of the critical temperatures given by Eq. (31) as a function of the deformation parameters p, q such that $0 < p \leq 1$ and $0 < q \leq 1$. *Right:* The ratio $T_c^{(r,\theta)}/T_c$ versus the deformation parameters r and θ , $0 < r \leq 1$, $0 \leq \theta \leq 2\pi$.

5 Deformed Bose gas of Tamm-Dancoff type

Let us consider thermodynamics of the q -Bose gas of Tamm-Dancoff (TD) type. Recall that the defining (along with (1)) commutation relation

$$aa^\dagger - qa^\dagger a = q^N \quad (32)$$

for the TD q -bosons, and the q -bracket of TD type

$$[X]_{TD} = Xq^{X-1} \quad (33)$$

stem from those of the two-parameter p, q -deformed model if one puts $p=q$, see (7). We assume that $0 < q \leq 1$ in the case of TD bosons. For some exotic properties of the TD type deformed oscillator see Ref. [29].

Performing when $\lambda^3/v \ll 1$ same calculations as in Sec. 3, in the Tamm-Dancoff case we find the coefficients for the virial expansion (24):

$$\begin{aligned} A_{TD} &= -\frac{q}{2^{5/2}} = qA_0, & C_{TD} &= q^3 \left(\frac{1}{2^{3/2}3^{1/2}} - \frac{3}{2^5} - \frac{5}{2^{11/2}} \right) = q^3 C_0, \\ B_{TD} &= q^2 \left(\frac{1}{2^3} - \frac{2}{3^{5/2}} \right) = q^2 B_0, & D_{TD} &= q^4 \left(\frac{7}{2^6} - \frac{1}{2^2 \cdot 3^{3/2}} + \frac{1}{2^{5/2}} + \frac{2}{3^3} - \frac{4}{5^{5/2}} \right) = q^4 D_0. \end{aligned}$$

It is worth to note that, due to $q \leq 1$, the TD Bose gas virial coefficients are lowered with respect to non-deformed virial coefficients, by the factor of q^{k-1} in the k -th virial coefficient. Accordingly, this weakens in q^{k-1} times the usual k -particle effective interaction within the TD q -Bose gas.

In the case of low temperatures and large densities, $\lambda^3/v \gg 1$, the critical temperature of Bose gas of Tamm-Dancoff type is

$$\frac{T_c^{TD}}{T_c} = \left(\frac{2.61}{g_{3/2}^{TD}(1; q)} \right)^{2/3}, \quad (34)$$

where $g_{3/2}^{TD}(1; q)$ is the TD analog of function (19) at $z=1$ and $n=3/2$. In general,

$$g_n^{TD}(z; q) = \sum_{r=1}^{\infty} \frac{q^{r-1} z^r}{r^n}. \quad (35)$$

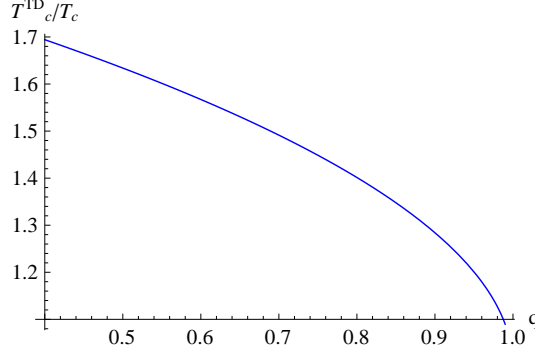


Figure 4: The ratio T_c^{TD}/T_c of the TD -type deformed critical temperature T_c^{TD} to the non-deformed one T_c versus the deformation parameter q where $0 < q \leq 1$.

In Fig. 4, we plot the ratio (34) versus deformation parameter q for $0 < q \leq 1$. As seen, the larger is deformation (deviation $1 - q$ from the non-deformed value $q = 1$), the larger is the critical temperature T_c^{TD} of deformed bosons of TD type. At $q = 1$ the ratio (34) gives 1, as it should be.

6 Concluding remarks

In this paper we developed some version of the p, q -deformed analog of the Bose gas model (p, q -Bose gas model). We studied the thermodynamics of such a gas at high (low) temperatures and low (large) densities. To obtain the thermodynamical quantities we utilized the extension of q -calculus based on the direct p, q -generalization of the Jackson derivative.

For high temperatures ($\lambda^3/v \ll 1$), dealing with the equation of state we have obtained the virial expansion and correspondingly derived the nontrivial virial coefficients A, B, C, D all depending on the deformation parameters p and q . Note that these parameters are contained in the virial coefficients through the p, q -numbers only and are thus symmetric under the exchange $p \leftrightarrow q$. In the limit $p = q = 1$ the virial coefficients A_0, B_0, C_0, D_0 of usual Bose gas are recovered. The parameters p and q are taken first to be real, and then also as complex valued such that $p = \bar{q} = r e^{i\theta}$. In these new variables r and θ we analyzed the virial coefficients as well. The differences between deformed coefficients and their non-deformed counterparts have been studied from the viewpoint of their dependence on the parameter θ . Figures 1, 2 demonstrate: there exist some special value(s) of the θ -parameter (and r) for which any chosen difference from (26) can vanish due to some mutual compensation. On the other hand, for special fixed value θ (at certain r), the virial coefficient \tilde{A} , and thus the effective two-particle interaction can be vanishing. Note that the effective two-particle interaction (and vanishing of it) means superimposing (and mutual compensation) of the conventional two-particle interaction, present for the usual non-deformed non-ideal Bose-gas, and the additional interaction imported due to the deformation used by us. Evidently, the same can be said about each from the rest of virial coefficients (taken alone).

At low temperatures ($\lambda^3/v \gg 1$) we find both the critical temperature $T_c^{(p,q)}$ of deformed p, q -Bose gas and the explicit dependence of the ratio $T_c^{(p,q)}/T_c$ on the parameters p and q (either taken to be real, or as the pair of mutually conjugate complex values). Similar results are deduced for the deformed Bose gas of Tamm-Dancoff type. We have found, for

the whole range of the considered p, q , that $T_c^{(p,q)} > T_c$ (such inequality retains in the case of TD-type q -Bose gas also). We hope this feature may have interesting consequences. To conclude, the issue of Bose condensation in its p, q -deformed manifestation (with $T_c^{(p,q)}$ or $T_c^{(r,\theta)}$ higher than T_c) is reproduced and pictured in Fig. 3, *left* and *right*. It would be interesting to make comparison of the obtained results with the critical T_c of a real gas.

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